of which the first is Wilson's Theorem, while the left member of the second is the coefficient of $x^m y^{p-1-m}$ in $(x+y)^{p-1}$. The latter is congruent, modulo p, to

$$\frac{x^p+y^p}{x+y}=x^{p-1}-x^{p-2}y+\ldots\ldots+(-1)^m x^m y^{p-1-m}+\ldots\ldots+y^{p-1}.$$

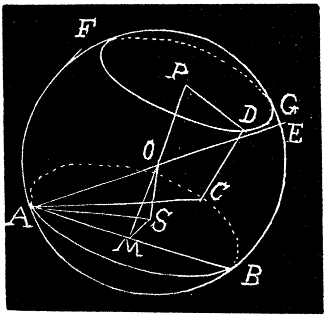
AVERAGE AND PROBABILITY.

130. Proposed by LON C. WALKER. A. M., Graduate Student. Leland Stanford University, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of a tetrahedron formed by the planes passing through the points taken three and three, is 1-35 of the volume of the given sphere.

I. Solution by the PROPOSER.

Choose A, B, C, D as the four random points; O the center of the given sphere with radius r: $ABFE$ a great circle through A, B; ABC a small circle through A, B, C, with center S; DGF a small circle through D parallel to ABC, with center P; M the middle point of AB.



Put $OP=x$, $AS=r_1$, $\angle AOB=\theta$, $\angle OMS=\phi$, $\angle CAB=\psi$, $\angle SAM=\psi_1$. Then we have

$AM=r\sin\frac{1}{2}\theta=r_1\cos\psi_1$,

$SM=r\cos\frac{1}{2}\theta\cos\phi=r_1\sin\psi_1$,

$OS=r\cos\frac{1}{2}\theta\sin\phi$,

$AC=2r_1\cos(\psi-\psi_1)$,

$PD=\sqrt{(r^2-x^2)}$,

$r_1=r(\sin^2\frac{1}{2}\theta+\cos^2\frac{1}{2}\theta\cos^2\phi)^{\frac{1}{2}}$, area $ABC=2rr_1\sin\frac{1}{2}\theta\sin\psi\cos(\psi-\psi_1)$,

volume of tetrahedron $D-ABC=\frac{1}{3}SP.\text{area}ABC=\frac{2}{3}rr_1(x+r\cos\frac{1}{2}\theta\sin\phi)\sin\frac{1}{2}\theta \times\sin\theta\cos(\psi-\psi_1)$.

Hence, we have for the required average volume, $V=\frac{4}{105}\pi r^3=\frac{1}{35}$ of the volume of the given sphere.

[For the integration, see solution in last issue, page 113, where the figure for Professor Zerr's solution was inserted for the one belonging to Professor Walker's solution. F.]

131. Proposed by LON C. WALKER. A. M., Graduate Student, Leland Stanford Jr. University. Cal.

A sphere is described with its center within a given sphere, and its surface intersecting the surface of the given sphere. The average volume common to both spheres is 10/21 of the volume of the given sphere.

Solution by the PROPOSER.

Let M be the center of the given sphere, and N that of the random sphere. Put $BM=a$, $NB=x$, $MN=y$, $\angle MBN=\theta$, $\angle BMN=\phi$, $\angle BNM=\psi$.

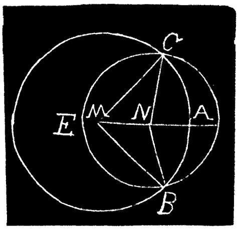
Then we have

$$\cos\phi=\frac{1}{y}(a-x\cos\theta),$$

$$\cos\psi=\frac{1}{y}(x-a\cos\theta),$$

$$y^2=a^2+x^2-2ax\cos\theta.$$

Volume of spherical section $BACM=\frac{2}{3}\pi a^3(1-\cos\phi)=\frac{2}{3}\pi[a^3-1/y(a^4-a^3\cos\theta)]$; volume of spherical sector $BECN=\frac{2}{3}\pi x^3(1-\cos\psi)=\frac{2}{3}\pi[x^3-1/y(x^4-ax^3\cos\theta)]$; volume of solid $BMCN=\frac{1}{3}\pi a^2y\sin^2\phi=\frac{1}{3}\pi(a^2x^2/y)\sin^2\theta$; volume of solid $BACE=S=\frac{1}{3}\pi[2a^3+2x^3-2/y\ (a^4+x^4)+2/y\ (a^3x+ax^3)\cos\theta-(a^2x^2/y)\sin^2\theta]$.



Hence we have for the required average volume,

$$V=\frac{\int_0^a\int_{a-x}^a S.4\pi y^2dxdy+\int_a^{2a}\int_{x-a}^a S.4\pi y^2dxdy}{\int_0^a\int_{a-x}^a 4\pi y^2dxdy+\int_a^{2a}\int_{x-a}^a 4\pi y^2dxdy}$$

$$=\frac{2\pi}{3a^4}\Bigg\{\int_0^a\int_{a-x}^a\Big[2y^2(a^3+x^3)-2y(a^4+x^4)+2y(a^3x+ax^3)\cos\theta$$

$$-a^2x^2y\sin^2\theta\Big]dxdy$$

$$+\int_0^{2a}\int_{x-a}^a\Big[2y^2(a^3+x^3)-2y(a^4+x^4)+2y(a^3x+ax^3)\cos\theta$$

$$-a^2x^2y\sin^2\theta\Big]dx\,dy\Bigg\}$$

$$=\frac{2\pi}{3a^4}\Bigg\{\int_0^{2a}\Big[\tfrac{2}{3}a^6-2a^5x+2a^4x^2+\tfrac{5}{4}a^2x^4-2ax^5+\tfrac{17}{24}x^6\Big]dx$$

$$+\int_0^a\tfrac{2}{3}(a^3+x^3)(x-a)^3dx+\int_a^{2a}\tfrac{2}{3}(a^3+x^3)(a-x)^3dx\Bigg\}$$

$=\frac{40}{63}\pi a^3=\frac{10}{21}$ of the volume of the given sphere.

Also solved by *G. B. M. ZERR*.

132. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics. The Temple College, Philadelphia, Pa.

n points are taken at random on the circumference of a given circle. Prove that the chance of the center of the circle lying within the polygon formed by joining these points is $1-(1/2^{n-2})$.